THE
MATHEMATICAL GAZETTE.

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THE MATHEMATICAL ASSOCIATION.

LONDON BRANCH ANNUAL MEETING.

THE first annual meeting of the London Branch was held at the London Day Training College on February 11th, when there was a large attendance of members. The officers of the Branch were re-elected, and the following new members of the Committee were selected in place of those who retired: Miss Punnett, Miss Raymond, Prof. P. Harding, and Mr. W. J. Dobbs.

Sir George Greenhill was nominated by the Committee as the first President of the Branch. After his Presidential Address, which appears below, a paper was read by Mr. C. S. Jackson on "The Introduction of the Calculus." In the discussion which followed contributions were made by W. J. Dobbs, W. D. Eggar, G. St. L. Carson, A. Lodge, A. W. Siddons, and L. Isserlis.

ANNUAL REPORT OF THE BRANCH, 1910.

THE Branch has made good progress during this—the first—year of its existence. It numbers 260 members, 136 full members, and 124 associates. Of the full members between 30 and 40 joined the Association in consequence of the foundation of the Branch. It is desired to increase still further the strength and influence of the Branch so that it may become a potent factor in the improvement and progress of mathematical teaching in London.

Five meetings have been held during the year at which papers have been read on the following subjects:

The Board of Education Circular on the Teaching of Geometry—Mr. A. Siddons.

Elementary Arithmetic—Mr. Grant and Mr. Whipple.

Graphs—Dr. J. P. Nunn and Mr. P. Abbott.

Irrational Numbers—Mr. G. St. L. Carson.

Teaching of Specific Volume—Mr. G. F. Daniell.

The attendance at these meetings has been very encouraging.

In addition to the meetings for the reading of papers, a Social Evening was held on December 10th at the Mercer's School, by the kind permission of the Mercer's Company and the Head Master, the Rev. Dr. Scott. At this meeting Professor Harding showed a most interesting set of slides dealing with the history of arithmetic, and there was a large exhibition of models and books.

The thanks of the Branch are due to all those who have contributed to the success of these meetings and to the authorities of the London Day Training College and the Polytechnic, Regent Street, who kindly lent the various rooms in which the meetings have been held.

Suggestions for the programme for the Session 1911-12 will be welcomed by the Committee.

NORTH WALES BRANCH.

A MEETING of this Branch was held on May 27, at the new University Buildings, Bangor, where Professor Bryan entertained the members present. Mr. R. J. Owen, of Holywell County School, read a paper on "Contracted Multiplication and Division in Arithmetic," in which he explained and compared the various methods employed in different text-books. In the ensuing discussion members referred to the special difficulties they meet with in school classes, and stated their preference for those methods which are not too technical. Prof. Bryan explained the Indian method of multiplication.

It was decided that 10s. out of surplus funds should be sent to the Treasurer of the Mathematical Association.

PRESIDENTIAL ADDRESS TO THE LONDON BRANCH OF THE MATHEMATICAL ASSOCIATION.

By SIR GEORGE GREENHILL, February 11th, 1911.

WHEN your chairman, Mr. Jackson, asked me to supply a title for the Presidential Address, I could think of no name short enough and suitable for these random remarks, *à bâtons rompus* for half an hour.

Professor Turner gave the Mathematical Association an account of some new methods in calculation, found successful for Halley's Comet; and Ray Lankester held forth in the afternoon on his perennial subject of the place of Greek in education, about which Turner has given you his say on a previous occasion.

I must avow myself at the outset in complete disagreement with their opinion, as in Greek we come to the bedrock of all our knowledge in history, literature, science, and art. Greek is the lighthouse to guide us out of barbarism to the human ideal.

In an earlier age, when there was little outlook for learning in this country, the classics could be taught leisurely to a select few, chiefly ecclesiastics, and a large grammatical foundation was laid for an intellectual edifice that seldom rose above the ground.

But methods are now in operation to show how small a grammatical drill, that need not hamper progress at the outset, will enable the modern student to lose none of the intellectual pleasures of his ancestors.

Here is the *Griechisches Lesebuch* of Wilamowitz-Moellendorf, intended expressly to interest the scientific student in the Greek writers, and for the subject-matter, not the mere grammatical parsing dear to the philologue.

An extract from Euclid is given in the *Lesebuch*, and our motto should be: "Euclid in Greek and Greek in Euclid," if only for the etymology of our terms in geometry.

The student of Simpson, Potts, and Todhunter will be surprised at the closeness of the treatment, word for word; and where any restoration has been attempted by the moderns, we prefer now the original.

So, too, we deplore the Victorian fake of all our ancient historical architecture, leaving nothing untouched.

The diagrams here of the Pneumatics of Hero of Alexandria—the Fire-Engine, for instance—are superior to those in the little text-books in use in

my time at Cambridge; and the precursor of the Parson's Steam Turbine can be shown at work, blown in glass, and held in a spirit lamp.

When I was in Berlin about a year ago, I showed this experiment off with great success at a dinner party given by Prof. Schwarz to our faculty, handing round first the description of Hero himself, in the extract given in the *Lesebuch*, and then the lamp and glass globe in action like fury.

I felt safe in offering this experiment, as Schwarz had let me know beforehand that he intended to improve the occasion with a history of the Hodograph, supported by references, showing that Hamilton himself, the reputed inventor, was quoting, as known, the theorem of the Hodograph of a Planetary Orbit, and thence the resolution of the velocity into two constant components, one perpendicular to the apsidal line, the other to the radius vector.

This gave me the opportunity for a mathematical impromptu—that in walking from our Hotel Furstenhof, and crossing the road in front of a tramcar, Prof. Klein and I had described by instinct a relative orbit round the tram driver in true planetary style, watching him nervously, and walking at right angles with constant velocity.

"So," said Schwarz, "it is the Fear which is inversely as the square of the distance—die Furcht is als die Entfernung quadrat umgekehrt—the planet going in fear of the Sun!"

It is a calamity if historical continuity should hold no place in our scientific ideas; and it was an evil day when Euclid was thrown down, carrying Newton with him in his fall.

Legendre replaces Euclid in France and America, although no better; and historical continuity is lost to the extent that the name, *pons asinorum*, is applied by them to Euc. i. 47, discovered by Pythagoras according to the legend.

I need not remind this learned company that Euc. i. 5 did not receive this name from any extraordinary difficulty, but because the figure resembles a trestle bridge with a ramp so steep as to be negotiable only by a donkey or a mule, not a horse; a *pons* not *equorum*, but *asinorum*.

Dulcarnon is the name given by Chaucer's Cressida (*T. and C.* Book iii. l. 931), and I asked Canon Wilson after his delightful historical address if he could explain the application. It may be from the Arabic equivalent for the Bride's Chair, the name revived by Clifford, and Skeat's edition of Chaucer (Oxford, 1894) has an interesting note on these words, with references to Roger Bacon.

The geometry which is to replace Euclid is not much more than the use of instruments on the drawing-board. Called Graphike (not Geometria), it was the first subject the little Greek boy was taught long before being promoted to Euclid and strict logical exposition and dialectic.

But Euclid is condemned now as tedious and prolix, and argument usually from an incorrect figure.

The modern method on the other hand is inarticulate and can only point to the diagram of instrumental accuracy and say, as an American: "And what is the matter with that?" The retort in Latin would be:

"Quodcunque ostendis mihi sic, *inarticulus*, odi!"

quoted, it is said, by the sea-sick scholar on the Ostend packet.

Newton's preface to his *Principia* discusses the same question—

"Rectas et Circulos describere problemata sunt, sed non geometrica.

"Ex Mechanica postulatur horum solutio, in Geometria docetur solutorum usus.

"Ac gloriatur Geometria quod tam paucis principiis aliunde petitis tam multa praestet."

Poincaré's address to the Mathematical Congress at Rome gave an instance of the French neglect of historical interest, when he classed among fables the story of Newton and the apple.

When I found him alone, I asked him why? We are indebted for the fable to Voltaire, who had it direct from Mrs. Conduitt, Newton's niece. "Newton, et la nièce de Newton, et Volt{aire / erra}," said Poincaré, "Non! Je ne vous comprends pas!"

It struck me afterwards that when I said Voltaire, he thought I said Volterra. The two names sound exactly alike.

Our chairman, Mr. Jackson, will remember the visit of Dr. Witting, of Dresden, to consult the Newton MSS. at Cambridge.

We showed him Plutarch's Symposium here, on the "Face in the Orb of the Moon," where the theory of Gravitation is enunciated, and Newton's First Law of Motion is given in the Greek equivalent of his own Latin statement.

We may imagine that Newton was studying it as the set book for the Compulsory Greek in the Little-Go, in the orchard at home, sent down from Cambridge by reason of the plague; and the studious youth would think at first he was being pelted by some rascal boys when he heard the thud of an apple; looked up and wondered as here in Plutarch, how the Aethiopian vertically below did not go in fear of its falling.

And Shakespeare has been there too, in Plutarch, to judge from Holofernes in *Love's Labour's Lost*.

"Now hangeth like a jewel in the ear of *coelo*, the sky, the welkin, the heaven; and anon falleth like a crab on the face of *terra*, the soil, the land, the earth."

And if Newton had read further in Plutarch, or in this extract from Archimedes' *Psammites*, he would have learned that the earth is 216 thousand stadium round, 21,600 miles in the Opus Majus, but in the geographical mile of the sailor, and not the soldier's land-mile of 1000 paces; and so Newton would have saved twenty years' delay in his theory.

Archimedes is here trying to make game of Aristarchus, for saying that the distance of a fixed star is as many times the distance of the sun as the radius, or surface, of the earth to that of its centre; which is absurd, Archimedes says; but Aristarchus meant that the fixed stars are at infinite distance.

The French (and the Americans too) date the history of the world from their Revolution, and the Metric System, its principal achievement, has made havoc with the ancient measures. The sailor, however, still employs the stadium as his cable length, and ten cable or stadium make the geographical sea-mile, thirty to the league (or parasang of Xenophon).

The sailor is not disposed to throw his chronometers into the sea for the sake of a scientific fad. But working with the sexagesimal time and angle of the Chaldaean he can start the decimal system on the geographical mile, or the minute of arc on the surface of the Earth, by dividing it up into a thousand Greek fathom.

Here, too, we find in Archimedes the index notation for recording high numbers, brought out recently as a great novelty.

The Greek boy was taught to do prodigies of calculation on his fingers; and I noticed that the calculator Inaudi kept his hands behind his back, and so (probably) had learned the Greek methods.

These should find a place in our Arithmetical instruction as developing instinctive rapidity, reserving for Algebra the logical reasoning to trip him up with stumbling-blocks, by making him think of the reason of every step. The modern method of mathematics strews the path of the novice with boulders.

Arithmetic and Analysis was classed as a pastime, but Geometry was compulsory in the Greek university, as shown by the inscription over the entrance to Plato's academy: ΟΥΔΕΙΣ ΑΓΕΩΜΕΤΡΗΤΟΣ ΕΙΣΙΤΩ.

But the option of Algebra is allowed at Oxford, and all take it, acting on the modern heuristic method of the Research of the Soft Option.

The difficulty about Greek at the present day is that so many students are encouraged by soft options to chance it, and if the compulsion should arise, they trust to cramming, and here they acquire the mental indigestion of a lifetime. But a boy should start Greek with Latin, or even earlier.

I remember my own intellectual pride at learning a new Alpha-Beta; and Greek struck me as a real language like English, with the little words, "he," "she," "it," "of," "will."

I was quite sure that no one ever talked the Latin we read.

Quaker Bright described Greek as a dead and heathen language.

But why heathen, when half our sacred books are in the original Greek?

And as to being dead, there is more Greek spoken in the Levant at the present day than all the Latin languages of the Ponent.

Naval and military education is purely utilitarian, and Greek is excluded; but as naval developments are more likely to arise in the Levant than elsewhere, a knowledge of Greek is of greater utility to a naval officer than French or German.

I could tell a yarn of how a change of plan saved us once from shipwreck inevitable if our Greek captain had followed the admiralty chart, which showed that the hydrographer was totally innocent of Greek.

We of our science should despise the bait of a soft option, and should take a delight in tracing the etymology of the mathematical vocabulary, such as, to quote only a few: mathematics, geometry, astronomy, mechanics, dynamics, statics, hydraulics, pneumatics, optics, axiom, parallel, rhombus, cube, sphere, cylinder, cone, parabola, ellipse, hyperbola, asymptote, pentagon, dodecahedron, pyramid, prism, horizon, zodiac, ecliptic, cycle, epoch, parallax, anomaly. A complete list would include most of the terms of our subject.

The opponents of Greek are making a desperate attack at the time when cheap printing and improved methods will meet their real objections.

They tell us what will happen, when the evidence of the facts is to the contrary.

I have here some recent French newspapers deploring the inarticulate results of the alterations already adopted in France, but ignored by our reformers, who take no trouble to find out the facts.

Mr. Harding's lecture at our last meeting was very entertaining—on Ancient Methods in Arithmetic, meaning, 200 years old.

I was tempted to join in the discussion with a story Routh used to tell of his experience as a London university examiner, which required him to set a paper on simple arithmetic and algebra.

So like an examiner, he must try to puzzle the candidate with an unfamiliar statement; and so he asked: "How many times can you subtract 17 from a million?"

A simple division, you see, "but he said SUBTRACT!" the defeated candidate complained.

Routh confessed, however, that one candidate scored off him. It wrote down a million at the top of the page, one and six noughts, 17 underneath in the right place, drew a line, subtracted, 389,999 in the inferior inverse order as we are taught and cannot unlearn, drew a double line, and ticked it off as correct. Then it went at it again, wrote down a million again, 17 underneath, subtracted; then again, till the page was full. Then it wrote at the foot: "I can do that as often as you like."

I had a similar experience myself when I gave the question: "Add up all the numbers from one to a million." Half a million times a million and one; five hundred thousand million five hundred thousand.

"Didn't you do that? Why, it is the sum of an arithmetical progression of all the numbers from one to a million"; that is the way I ought to have set the question. "Oh! if you had only said that, I could have done it."

But scores of the candidates started writing 1, 2, 3, 4, ... in a vertical column, added them up, carried over to the next page, and so on till they were tired, and wrote the time-honoured formula—TIME. If the candidate had added: "and at this rate it would take me 30 years!" I think it would have deserved full marks.

Difference in Similarity and Similarity in Difference has been called the motto of our Science—or words to that effect. A Similarity in Difference halves the subject at once, like Polar Reciprocals, or Projection in Geometry, and Division and Subtraction in Arithmetic.

I may conclude with an illustration out of Analysis.

When Sylvester was invited to the Chair at Oxford about 1884, he was full of the subject of the Reciprocant, now clean forgotten, although at the time he enlisted us all in the interest.

One day after, Sylvester was noticed walking alone, addressing the sky, asking it: "Are Reciprocants all Bosh? Berry of King's says the Reciprocant is all Bosh!" There was no reply, and Sylvester himself was tiring of the subject, and so Berry escaped a castigation.

But recently I had occasion from the Aeronautical point of view to work out the theory of a Vortex inside a Polygon, an eddy whirlwind such as Chavez had to encounter in the angle of the precipices, flying over the Simplon Pass.

The analysis in some cases seemed strangely familiar, and at last I recognised the familiar Reciprocant.

ON TERTIAL, QUINTAL, ETC., FRACTIONS.

(*Continued from page* 67.)

8. *Conjugate Half-Periods.* When ξ is *even*, the period of ξ digits may be divided into two half-periods of equal length $=\frac{1}{2}\xi$. In this case let the period be

$$1/N=a_1a_2a_3\ldots a_{\frac{1}{2}\xi}a_{\frac{1}{2}\xi+1}a_{\frac{1}{2}\xi+2}\ldots a_\xi \ [\xi \textit{ even}].$$

Then in certain cases the two half-periods are *conjugate*, digit by digit, in such a way that

$$a_1+a_{\frac{1}{2}\xi+1}=a_2+a_{\frac{1}{2}\xi+2}=\ldots=a_{\frac{1}{2}\xi}+a_\xi=(r-1)=q. \quad \ldots\ldots\ldots\ldots(16)$$

Hence, if P', P'' be the two half-periods of $\frac{1}{2}\xi$ digits each,

$$P'+P''=qqq\ldots q=(r^{\frac{1}{2}\xi}-1) \quad \ldots\ldots\ldots\ldots(16a)$$

Hence also, if P' begins with α ciphers, then P'' begins with α digits $=q$. ...(17)

It will be found the half-periods are,—(provided ξ be *even*)—

Conjugate. When $N=p,\ p^2,\ p^3$, etc.; $2p,\ 2p^2$, etc.; (r^a+1). $\ldots\ldots\ldots\ldots(17a)$

Non-conjugate. Usually when $N=4p,\ 4p^2$, etc.; $8p$, etc.; $p_1p_2,\ 2p_1p_2$, etc., (17*b*)

where p denotes a prime, and p_1, p_2 are different primes. The Tables at the end supply numerous examples: a circumflex ($\wedge$) is there used to show the point of separation of the Half-Periods.

There are exceptions* to the above Rule (17*b*) as to Non-conjugation.

Ex. $r=3$; $N=28$, 76 give conjugate half-periods.
$r=5$; $N=21$, 69, 87; 42 give conjugate half-periods.

Also, $N=(r^a+1)$ always gives conjugate half-periods. $\ldots\ldots\ldots\ldots(17a)$

9. *Conjugate Endings.* $N=(r^a\mp X)$. $[X<r^a.]$

If $N'=r^a-X$, $N''=r^a+X$, $[X<r^a]$,

*The general Rule as to Non-conjugate Half-Periods has not yet been found.

and if the a digits at the end of $1/N'$, $1/N''$ be

$$\ldots a'_a, a'_{a-1}, \ldots a'_2, a'_1 \text{ of } 1/N'; \quad \ldots a''_a, a''_{a-1}, \ldots a''_2, a''_1 \text{ of } 1/N'',$$

it will be found that the corresponding a digits at end of $1/N'$, $1/N''$ are *conjugate* as follows:

$$a'_a + a''_a = a'_{a-1} + a''_{a-1} = \ldots = a'_2 + a''_2 = r - 1, \quad \ldots\ldots\ldots\ldots (18a)$$

and lastly

$$a'_1 + a''_1 = r, \quad \ldots\ldots\ldots\ldots (18b)$$

but this property extends only to the a digits at the end of the periods. [See Table following.]

This property enables the end a figures of the reciprocals of whole groups of large numbers $N'' = (r^a + X)$ to be written down at sight when the endings of the reciprocals of $N' = (r^a - X)$ are known for a number of values of X.

Ex. The Tables at the end give the *complete periods* of $1/N$ from $N = 1$ to 100 in the 3- and 5-scales, so that writing

$$N' = 3^a - X \quad \text{or} \quad 5^a - X$$

the a end-figures (in the 3- and 5-scales) of the reciprocals of the 100 numbers N' in the range

$$X = (3^a - 1) \text{ to } (3^a - 100); \quad (5^a - 1) \text{ to } (5^a - 100)$$

are known. Hence, by the above property, the a figures at end of the reciprocals of the 100 numbers

$$N'' = (3^a + X) \quad \text{or} \quad (5^a + X)$$

are known (as conjugates of the former) for the same range of X.

10. *Tables of Tertial and Quintal Fractions.*

The Tables at end of this paper give the following data for the Tertial and Quintal Fraction of $1/N$ for all integers $N \not> 100$ to Argument N (excluding multiples of 3 and 5 from the Tertial and Quintal Tables respectively for the reason given in Art. 2*b*).

Col. 1 of N_x gives N in denary scale.

Col. 2 of N_{iii}, N_v gives N in the ternary and quinary scales respectively.

Col. 3 gives the *number* of the figures (n) in the Tertial or Quintal Fractions of $1/N_{iii}$ or $1/N_v$.

Col. 4 gives the Repetend Period (P) of the Tertial or Quintal Fraction $1/N_{iii}$ or $1/N_v$, properly *pointed* so as to show the Repetend clearly (Art. 1, 3°). When the Repetend consists of two *conjugate* Half-Periods (Art. 8) a circumflex (^) is placed between them to show the point of separation.

Errata in the previous part of this Paper (pages 63–67).

Page 63. Last line of footnote. *For* M. J. Woodall, *read* H. J. Woodall.

„ 64. In Result (1*b*). *For* $\frac{1}{r}$, *read* $\frac{1}{r^a}$.

„ „ „ (1*d*). *For* $\overline{r-1}$, *read* $\overline{r^3-1}$.

„ „ „ (1*i*). *For* q^5+1, *read* r^5+1.

CONJUGATE ENDINGS OF TERTIALS AND QUINTALS.

	$1 \div (81 \mp x)$	
x	$-x$	$+x$
1	000ī	222$\dot{2}$
2	111$\dot{2}$	111ī
4	202ī	020$\dot{2}$
5	210$\dot{2}$	012ī
7	201ī	021$\dot{2}$
8	212$\dot{2}$	010ī
10	220ī	002$\dot{2}$
11	201$\dot{2}$	020ī
13	022ī	200$\dot{2}$
14	100$\dot{2}$	122ī
16	221ī	001$\dot{2}$
17	202$\dot{2}$	020ī
19	210ī	012$\dot{2}$
20	221$\dot{2}$	001ī
22	211ī	010$\dot{2}$
23	220$\dot{2}$	002ī
25	011ī	211$\dot{2}$
26	122$\dot{2}$	100ī
28	200ī	022ī
29	011$\dot{2}$	211ī
31	102ī	120ī
32	110$\dot{2}$	112ī
34	101ī	121$\dot{2}$
35	112$\dot{2}$	110ī
37	120ī	102ī
38	101$\dot{2}$	121ī
40	222ī	000$\dot{2}$
41	000$\dot{2}$	222ī

	$1 \div (125 \mp x)$	
x	$-x$	$+x$
1	00ī	444
2	22$\dot{3}$	222
3	13$\dot{2}$	31$\dot{3}$
4	33$\dot{4}$	111
6	04ī	404
7	03$\dot{3}$	41$\dot{2}$
8	14$\dot{2}$	30$\dot{3}$
9	024	42ī
11	33ī	114
12	24$\dot{3}$	24$\dot{2}$
13	30$\dot{2}$	14$\dot{3}$
14	014	43ī
16	32ī	124
17	40$\dot{3}$	042
18	01$\dot{2}$	43$\dot{3}$
19	304	14ī
21	01ī	434
22	41$\dot{3}$	03$\dot{2}$
23	32$\dot{2}$	12$\dot{3}$
24	344	10ī
26	40ī	044
27	32$\dot{3}$	12$\dot{2}$
28	23$\dot{2}$	21$\dot{3}$
29	234	21ī
31	44ī	004

	$1 \div (125 \mp x)$	
x	$-x$	$+x$
32	13$\dot{3}$	31$\dot{2}$
33	24$\dot{2}$	20$\dot{3}$
34	42$\dot{4}$	02ī
36	23ī	214
37	34$\dot{3}$	10$\dot{2}$
38	40$\dot{2}$	04$\dot{3}$
39	414	03ī
41	22ī	224
42	04$\dot{3}$	44$\dot{2}$
43	11$\dot{2}$	33$\dot{3}$
44	204	24ī
46	41ī	034
47	41$\dot{3}$	43$\dot{2}$
48	42$\dot{2}$	02$\dot{3}$
49	244	20ī
51	30ī	144
52	42$\dot{3}$	022
53	33$\dot{2}$	11$\dot{3}$
54	134	41ī
56	34ī	104
57	23$\dot{3}$	21$\dot{2}$
58	34$\dot{2}$	10$\dot{3}$
59	324	12ī
61	12ī	314
62	44$\dot{3}$	002
63	002	44$\dot{3}$

TABLE OF TERTIAL FRACTIONS ($1/N$).

[n=number of figures in Repetend (P) of $1/N$.]

N_x	N_{iii}	n	Repetend (P) of Tertial Fraction ($1/N$).
2	2	1	·i
4	11	2	·02
5	12	4	·0121
7	21	6	·0102 12
8	22	2	·01
10	101	4	·0022
11	102	5	·0021 1
13	111	3	·002
14	112	6	·0012 21
16	121	4	·0012
17	122	16	·0011 2021 2211 0201
19	201	18	·0011 0210 0221 1201 22
20	202	4	·0011
22	211	5	·0010 2
23	212	11	·0010 1120 021
25	221	20	·0010 0201 1022 1220 2112
26	222	3	·001
28	1 001	6	·0002 22
29	1 002	28	·0002 2101 0201 1122 2001 2120 2111
31	1 011	30	·0002 1211 1221 0202 2201 0111 0012 02

TABLE OF TERTIAL FRACTIONS ($1/N$)—*continued.*

[n = number of figures in Repetend (P) of $1/N$.]

N_x	N_{III}	n	Repetend (P) of Tertial Fraction ($1/N$). 4 8 12 16 20 24 28 32 36 40 44 48 52 56 60 64 68 72 76 80 84 88
32	1 012	8	·0̇002 112i̇
34	1 021	16	·0̇002 1010 2220 121i̇
35	1 022	12	·0̇002 0221 110i̇
37	1 101	18	·0̇002 0120 0222 0210 2i̇
38	1 102	18	·0̇002 0101 1222 0212 1i̇
40	1 111	4	·0̇00i̇
41	1 112	8	·0̇001 222i̇
43	1 121	42	·0̇001 2122 1202 0021 1121 0222 1010 0102 0220 1110 1i̇
44	1 122	10	·0̇001 2112 0i̇
46	1 201	11	·0̇001 2021 12i̇
47	1 202	23	·0̇001 2011 1210 0202 2001 01i̇
49	1 211	42	·0̇001 1221 2200 2012 1112 0222 1100 1002 2021 0111 0i̇
50	1 212	20	·0̇001 1212 0122 2110 102i̇
52	1 221	6	·0̇001 1i̇
53	1 222	52	·0̇001 1120 2101 0120 0211 2202 2122 2111 0201 2121 0220 1100 200i̇
55	2 001	20	·0̇001 1102 0212 1200 122i̇
56	2 002	6	·0̇001 1i̇
58	2 011	28	·0̇001 1012 0100 2022 2112 1021 220i̇
59	2 012	29	·0̇001 1010 0121 1102 1022 2002 0212 i̇
61	2 021	10	·0̇001 0222 1i̇
62	2 022	30	·0̇001 0220 2110 1212 2212 0020 1121 0i̇

64	2 101	16	·0001 0210 1112 202
65	2 102	12	·0001 0201 221
67	2 112	22	·0001 0121 2202 2212 1010 02
68	2 111	16	·0001 0120 1110 022
70	2 121	12	·0001 0110 201
71	2 122	35	·0001 0102 1020 0020 2112 1100 1112 0012 201
73	2 201	12	·0001 0022 212
74	2 202	18	·0001 0021 1222 1220 11
76	2 211	18	·0001 0012 0222 1221 02
77	2 212	30	·0001 0011 0121 2111 0222 0220 0120 21
79	2 221	78	·0001 0002 0011 0022 0121 1012 2102 1212 0201 1102 2212 2202 2112 2002 1011 2100 1201 0102 0211 12
80	2 222	4	·0001
82	10 001	8	·0000 2222
83	10 002	41	·0000 2221 0010 2201 0120 1211 2022 1101 0020 2111 1
85	10 001	16	·0000 2212 0120 0202
86	10 012	42	·0000 2211 0212 1122 0210 1222 2001 1201 0110 0201 21
88	10 021	10	·0000 2202 12
89	10 022	88	·0000 2201 2011 0202 0001 2110 1022 1111 0010 1220 2121 2222 0021 0211 2020 2221 0112 1200 1111 2212 1002 0101
91	10 101	6	0000 22
92	10 102	22	·0000 2122 0221 1121 0102 11
94	10 111	23	·0000 2120 2101 1212 2112 002
95	10 112	36	·0000 2120 0012 0100 1010 2002 0211 0111 122
97	10 121	48	·0000 2111 1220 2022 1212 2210 2222 0111 1002 0200 1010 0012
98	10 122	42	·0000 2110 2211 2121 0202 1222 2011 2001 1010 1202 01
100	10 201	20	·0000 2102 1211 1020 0122

TABLE OF QUINTAL FRACTIONS ($1/N$).

[n = number of figures in Repetend (P) of $1/N$.]

N_x	N_v	n	Repetend (P) of Quintal Fraction ($1/N$). 4 8 12 16 20 24 28 32 36 40 44 48 52 56 60 64 68 72 76 80 84 88 92 96
2	2	1	·2
3	3	2	·1$\overset{\wedge}{3}$
4	4	1	·1
6	11	2	·0$\overset{\wedge}{4}$
7	12	6	·0$\overset{\wedge}{3}$24 12
8	13	2	·0$\overset{\wedge}{3}$
9	14	6	·0234$\overset{\wedge}{2}$1
11	21	5	·0$\overset{\wedge}{2}$11 4
12	22	2	·0$\overset{\wedge}{2}$
13	23	4	·01$\overset{\wedge}{4}$3
14	24	6	·01$\overset{\wedge}{3}$4 3$\overset{\wedge}{1}$
16	31	4	·01$\overset{\wedge}{2}$4
17	32	16	·0121 3402 4323 1042
18	33	6	·011$\overset{\wedge}{4}$ 3$\overset{\wedge}{3}$
19	34	9	·011$\overset{\wedge}{2}$ 4214 1
21	41	6	·010$\overset{\wedge}{4}$ 3$\overset{\wedge}{4}$
22	42	5	·010$\overset{\wedge}{3}$ 2
23	43	22	·0102 0413 321$\overset{\wedge}{4}$ 3424 0311 2$\overset{\wedge}{3}$
24	44	2	·0$\overset{\wedge}{1}$
26	101	4	·0$\overset{\wedge}{0}$44
27	102	18	·00$\overset{\wedge}{4}$3 0332 244$\overset{\wedge}{0}$ 1411 2$\overset{\wedge}{2}$
28	103	6	·00$\overset{\wedge}{4}$2 1$\overset{\wedge}{3}$
29	104	14	·0041 2334 40$\overset{\wedge}{3}$2 1$\overset{\wedge}{1}$
31	111	3	·00$\overset{\wedge}{4}$
32	112	8	·0034 231$\overset{\wedge}{2}$
33	113	10	·0033 4322 03
34	114	16	·0032 1423 4411 302$\overset{\wedge}{1}$

32	112	8	·0034 2312
33	113	10	·0033 4322 03
34	114	16	·0033 1423 4411 3021
36	121	6	·0032 14
37	122	36	·0031 4212 2040 1133 4244 1302 3224 0433 1102
38	123	9	·0031 2104 3
39	124	4	·0031
41	131	20	·0030 1102 2044 1433 4224
42	132	6	·0024 42
43	133	42	·0024 2314 1223 4340 4311 1442 0213 0322 1010 4013 33
44	134	5	·0024 1
46	141	22	·0023 2431 4104 4212 0130 34
47	142	46	·0023 1221 0411 1424 1401 0124 4213 2234 0333 0203 0434 32
48	143	4	·0023
49	144	42	·0022 3341 4321 3314 0424 3442 2110 3012 3113 0402 01
51	201	16	·0022 1114 1241 0144
52	202	4	·0022
53	203	52	·0021 3440 1201 4204 1103 1223 3144 2310 0432 4302 4033 4132 2113
54	204	18	·0021 2413 3442 3204 11
56	211	6	·0021 04
57	212	18	·0020 4403 0133 4122 12
58	213	14	·0020 3414 4241 03
59	214	29	·0020 2440 3401 3214 1301 1134 2112 1
61	221	30	·0020 1103 3210 1304 4243 3411 2343 14
62	222	3	·002
63	223	6	·0014 43
64	224	16	·0014 3403 2242 1131
66	231	10	·0014 2133 24
67	232	22	·0014 1310 1024 4303 1343 42
68	233	16	·0014 0434 2203 1233

TABLE OF QUINTAL FRACTIONS ($1/N$)—*continued.*

[n = number of figures in Repetend (P) of $1/N$.]

N_x	N_v	n	Repetend (P) of Quintal Fraction ($1/N$). 4 8 12 16 20 24 28 32 36 40 44 48 52 56 60 64 68 72 76 80 84 88 92 96
69	234	22	·0014 0121 1034 4304 3233 41̇
71	241	5	·0013 4̇
72	242	6	·0013 32̇
73	243	72	·0013 2401 0032 0302 0114 1104 0233 2213 1021 4431 2043 4412 4142 4330 3340 3211 2231 342̇3
74	244	36	·0013 2103 3242 3041 4344 3123 4112 0214 030̇1
76	301	9	·0013 1024 4̇
77	302	30	·0013 0243 0112 2132 1004 4133 4034 22̇
78	303	4	·001̇3
79	304	39	·0012 4234 3022 3121 1440 1113 1032 2003 0402 41̇1
81	311	54	·0012 3242 2311 3433 4031 4414 2404 4321 2022 1331 0110 4130 0302 04̇
82	312	20	·0012 3023 3244 3214 211̇2
83	313	82	·0012 2311 1303 1240 3424 2121 0330 1400 4224 3344 1443 2213 3314 1320 4102 0232 3411 4304 4022 0110 03̇
84	314	6	·0012 2̇1
86	321	42	·0012 1132 0334 2142 4403 0443 2331 2411 0230 2004 1̇4
87	322	14	·0012 0424 4324 0̇2
88	323	10	·0012 0223 4̇3
89	324	44	·0012 0024 0103 0211 0422 1344 3244 2043 4142 3340 223̇1
91	331	12	·0011 4132 2424
92	332	22	·0011 3440 4302 2103 2312 4̇2
93	333	6	·0011 3̇3
94	334	46	·0011 3110 2430 3212 0423 0034 4331 3342 0141 2324 0214 4̇1
96	341	8	·0011 223̇4
97	342	96	·0011 2102 0123 3122 1411 4344 1031 3340 1400 2242 0403 0212 4433 2342 4321 1322 3033 0100 3413 1104 3044 2202 4041 423̇2
98	343	42	·0011 1420 4410 4132 0212 1443 3302 4003 4031 2423 2̇3
99	344	30	·0011 1240 3231 4244 2214 1324 3104 0̇1

ELLIPTIC TRAMMELS AND FAGNANO POINTS.

(*Continued from page* 78.)

Constructions for Fagnano Points.

22. To obtain a Fagnano pair of points on an ellipse whose axes are given, it may be considered sufficient to construct the corresponding Fagnano quadrilateral. That is, having drawn the line hk (Fig. 14) with the points p and z marked, and having described a circle on hk as diameter, it is only necessary to draw through z some chord c_1c_2 and to join the sides of the quadrilateral c_1hc_2k.

(i) If P_1 is given on the ellipse, c_1 is determined in Fig. 14. Hence to find P_2, the Fagnano conjugate of P_1, we must join c_1z and produce it to cut the circle on hk a second time in c_2. Then c_2hk is the trammel triangle of P_2.

(ii) To obtain a Fagnano pair of points having their common Fagnano distance equal to a given line less than $a-b$: describe a semicircle on pz (Fig. 14), and in it place the chord pm equal to the given line.

Let mz cut the circle on hk in c_1 and c_2. Then c_1hc_2k is the Fagnano quadrilateral required.

(iii) To obtain a Fagnano double point: draw c_1c_2 through z perpendicular to hk. Then the trammel triangles c_1hk and c_2hk will determine the same point on the ellipse.

(iv) If c_1c_2 coincides with hk, the vertices, A and B, of the ellipse are obtained as a Fagnano pair.

For other constructions see Art. 26 (i).

Properties of a Fagnano Pair of Points.

23. Since a Fagnano quadrilateral is a representative of a Fagnano pair of points, properties of such a quadrilateral, when translated back to the ellipse, will appear as properties of Fagnano points.

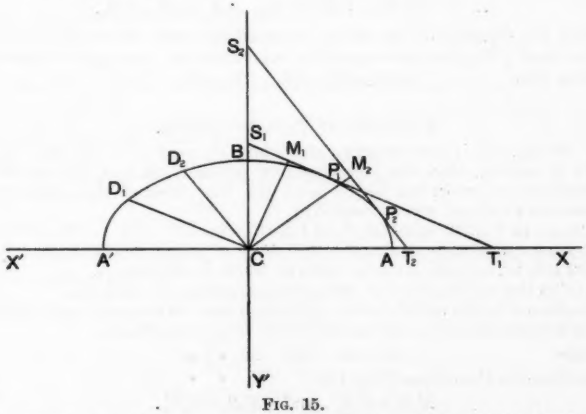


Fig. 15.

Fig. 15 shows P_1 and P_2, a Fagnano pair of points on an ellipse; S_1T_1 and S_2T_2 the tangents at P_1 and P_2; CD_1 and CD_2 the corresponding semi-conjugate diameters; and CM_1, CM_2 the perpendiculars from C on to the tangents.

Fig. 16 shows the corresponding Fagnano quadrilateral and its circumcircle. From the circle, we have (Fig. 14 or 16)

$$c_1z \,.\, c_2z = hz \,.\, kz = ab.$$

But $$c_1z = CD_1 \quad \text{and} \quad c_2z = CD_2$$

$$\therefore\ CD_1 \,.\, CD_2 = ab.$$

That is, the rectangle contained by semi-diameters conjugate to a Fagnano pair is constant.

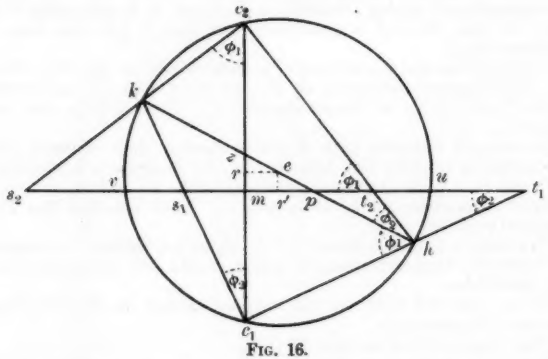


FIG. 16.

24. Let e be the centre of the circle in Fig. 16, and therefore the middle point both of hk and pz. Draw er perpendicular to c_1c_2. Then r is the middle point both of c_1c_2 and mz.

$$\therefore\ c_1m = c_2z \quad \text{and} \quad c_2m = c_1z.$$

$$\therefore \text{ in Fig. 15, } \quad CM_1 = CD_2 \quad \text{and} \quad CM_2 = CD_1.$$

That is, the distance of the centre of an ellipse from the tangent at one of the points of a Fagnano pair equals the semi-diameter conjugate to the other.

Hence also $$CM_1 \,.\, CM_2 = CD_2 \,.\, CD_1 = ab.$$

TANGENTS AT FAGNANO POINTS.

25. In Fig. 15, if the tangents at P_1 and P_2 meet CX in T_1 and T_2, and CY in S_1 and S_2, then the corresponding points t_1, t_2, s_1, s_2 in Fig. 16 will be obtained by producing pm to meet the four sides of the quadrilateral (themselves produced when necessary).

Thus s_1t_1 in Fig. 16 equals S_1T_1 in Fig. 15, etc.

Let pm cut the circle in u and v, and draw er' perpendicular to pm.

Then pm, uv, s_1t_2, s_2t_1 are four pairs of points in involution.

But r' is the middle point of each of the segments pm and uv.

Therefore r' is the middle point of the segment between any pair of points in this involution, or r' is the middle point of s_1t_2 and of s_2t_1.

Hence $$ms_1 = pt_2 \quad \text{and} \quad ms_2 = pt_1.$$

Therefore, in the ellipse (Fig. 15),

$$M_1S_1 = P_2T_2 \quad \text{and} \quad M_2S_2 = P_1T_1.$$

But by definition of P_1 and P_2 as a Fagnano pair, $P_1M_1 = P_2M_2$.

Hence the three segments S_1M_1, M_1P_1, P_1T_1 of the tangent S_1T_1 are respectively equal to the three segments P_2T_2, M_2P_2 and S_2M_2 of the tangent S_2T_2.

Properties of a Fagnano Pair of Points involving their Eccentric Angles.

26. Let ϕ_1 and ϕ_2 be the eccentric angles of the Fagnano pair P_1 and P_2; ϕ_1 is greater than ϕ_2, since P_1 is in BW, but P_2 in AW.

Then $c_1hk=\phi_1$ and $c_2hk=\phi_2$.

(i) $\therefore \tan\phi_1\tan\phi_2=\dfrac{c_1k}{c_1h}\cdot\dfrac{c_2k}{c_2h}$.

But $c_1k\,.\,c_2k=hk\times$ perp. from k on c_1c_2.

Also $c_1h\,.\,c_2h=hk\times$ perp. from h on c_1c_2.

$$\therefore \tan\phi_1\tan\phi_2=\frac{\text{perp. from } k \text{ on } c_1c_2}{\text{perp. from } h \text{ on } c_1c_2}=\frac{kz}{hz}=\frac{b}{a}.$$

Any two acute angles satisfying the last equation are eccentric angles of a Fagnano pair of points.

Hence a Fagnano pair may be obtained by taking, on the tangent to the ellipse at A, two points L_1 and L_2 such that $AL_1\,.\,AL_2=ab$ (Fig. 19).

Then CL_1 and CL_2 will cut the auxiliary circle in Q_1 and Q_2, points corresponding to a Fagnano pair on the ellipse.

The following construction also gives a Fagnano pair of points.

Take CQ_1 and CE, two radii of the auxiliary circle, so that the angles BCQ_1 and ACE are equal (Fig. 17).

Then

$$\tan ACQ_1\,.\,\tan ACE=1.$$

Let the ordinate of E cut the ellipse in F, and CF meet the auxiliary circle in Q_2.

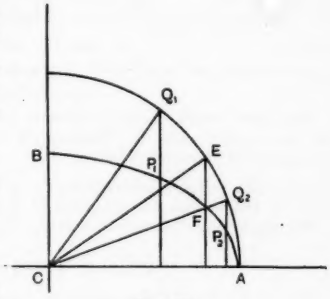


Fig. 17.

Then $\tan ACQ_2=\dfrac{b}{a}\tan ACE$, and hence $\tan ACQ_1\,.\,\tan ACQ_2=\dfrac{b}{a}$.

It follows that the ordinates of Q_1 and Q_2 will cut the ellipse in a Fagnano pair of points P_1 and P_2.

(ii) Since $$\sin c_1hc_2=\frac{c_1c_2}{\text{diam. of circle } c_1hc_2},$$

we have $$c_1c_2=hk\sin(\phi_1+\phi_2),$$

or, in the ellipse, $$CD_1+CD_2=(a+b)\sin(\phi_1+\phi_2).$$

(iii) If c_1e and c_2e are joined, then the angle $c_1ez=2\phi_1$ and angle $c_2ez=2\phi_2$.

But it is easy to see that

$$\frac{c_1z}{c_2z}=\frac{\sin c_1ez}{\sin c_2ez};$$

$$\therefore \frac{CD_1}{CD_2}=\frac{\sin 2\phi_1}{\sin 2\phi_2}.$$

But $$CD_1\,.\,CD_2=ab;$$

$$\therefore CD_1^2=\frac{ab\sin 2\phi_1}{\sin 2\phi_2}$$

and $$CD_2^2=\frac{ab\sin 2\phi_2}{\sin 2\phi_1}.$$

(iv) Since a circle circumscribes the quadrilateral c_1hc_2k, we have

$$\text{angle } c_1c_2k = \text{angle } c_1hk = \phi_1,$$

and similarly $$\text{angle } c_2c_1k = \phi_2.$$

But s_1t_1 and c_1t_1 are perpendiculars to c_1c_2 and c_1k;

$$\therefore \text{ angle } c_1t_1s_1 = \text{angle } c_2c_1k = \phi_2.$$

Similarly the angle $c_2t_2s_2 = \phi_1$.

Further, $$\text{angle } hpt_1 = \text{angle } c_1hp - \text{angle } ht_1p = \phi_1 - \phi_2;$$

$$\therefore \text{ angle } zpm = \phi_1 - \phi_2.$$

Hence, in the ellipse the inclination of the tangent to the trammel line at either of the points P_1 and P_2 is equal to $\phi_1 - \phi_2$.

(v) In Fig. 16, $$mz = pz \sin zpm.$$

But $$mz = c_1z - c_1m = c_1z - c_2z.$$

$$\therefore \text{ in the ellipse } CD_1 - CD_2 = (a-b)\sin(\phi_1 - \phi_2).$$

(vi) *Value of the common Fagnano distance of a Fagnano pair of points.*

In Fig. 16, $$pm = pz\cos zpm = (a-b)\cos(\phi_1 - \phi_2).$$

$\therefore$ in the ellipse the Fagnano distance of either of the points

$$P_1 \text{ and } P_2 = (a-b)\cos(\phi_1 - \phi_2).$$

Also, since $pm^2 = pz^2 - mz^2$, the square of this Fagnano distance

$$= (a-b)^2 - (CD_1 - CD_2)^2.$$

The last result is an extension to a Fagnano pair of a property given for one point only in Macaulay's *Geometrical Conics*. See Example 21, Chapter III., Second Edition.

(vii) The perpendicular from c_1 on to $hk = c_1z \,.\, \sin c_1zp = c_1z \,.\, \cos zpm$.

$\therefore$ in the ellipse the distance of C from $H_1K_1 = CD_1\cos(\phi_1 - \phi_2)$ (Fig. 12).

Similarly the distance of C from $H_2K_2 = CD_2\cos(\phi_1 - \phi_2)$.

Further Properties of Tangents.

27. (i) In Fig. 16 the angles pt_1h and pht_2 are each equal to ϕ_2;

$$\therefore \; pt_1 \,.\, pt_2 = ph^2;$$

$$\therefore \text{ in the ellipse (Fig. 15), } P_1T_1 \,.\, P_2T_2 = b^2.$$

But $$P_2T_2 = M_1S_1 \text{ (Art. 25)};$$

$$\therefore \; P_1T_1 \,.\, M_1S_1 = b^2.$$

Also the angles ps_2k and pks_1 are each equal to $\frac{\pi}{2} - \phi_1$;

$$\therefore \; ps_1 \,.\, ps_2 = pk^2;$$

$$\therefore \text{ in the ellipse } P_1S_1 \,.\, P_2S_2 = a^2.$$

But $$P_2S_2 = M_1T_1;$$

$$\therefore \; P_1S_1 \,.\, M_1T_1 = a^2.$$

(ii) The results just obtained are, in their final forms, concerned with only one tangent to an ellipse. Hence we see that, omitting the suffixes, if the tangent to an ellipse at any point P meets the axes CX and CY in T and S, and if CM is perpendicular to ST, $\therefore$

then $$PT \,.\, MS = b^2$$

and $$PS \,.\, MT = a^2.$$

But we have identically,

$$PS \,.\, MT = PT \,.\, MS + ST \,.\, PM;$$

$$\therefore \; ST \,.\, PM = a^2 - b^2.$$

From the last result it follows that ST will be least when PM is greatest; that is when P is the Fagnano double point.

In this case ST coincides with the trammel line through P and equals $a+b$.

(iii) In Art. 26 (vi) we found that

$$P_1M_1=(a-b)\cos(\phi_1-\phi_2).$$

But from (ii) of this article,

$$S_1T_1 \cdot P_1M_1=a^2-b^2;$$

$$\therefore\ S_1T_1 \text{ (or } S_2T_2)=(a+b)\sec(\phi_1-\phi_2).$$

PARALLELS AND PARALLELOGRAMS ARISING OUT OF FAGNANO POINTS.

28. In Fig. 18 is shown a pair of Fagnano points, P_1 and P_2, with the corresponding trammel lines H_1K_1, H_2K_2, and the tangents S_1T_1 and S_2T_2.

If ϕ_1 and ϕ_2 are the eccentric angles of P_1 and P_2, then the angle

$$CH_1K_1=\phi_1.$$

But in the corresponding figure of the Fagnano quadrilateral (Fig. 16), we have the angle $c_2t_2s_2=\phi_1$;

$$\therefore \text{ (in Fig. 18) the angle } CT_2S_2=\phi_1.$$

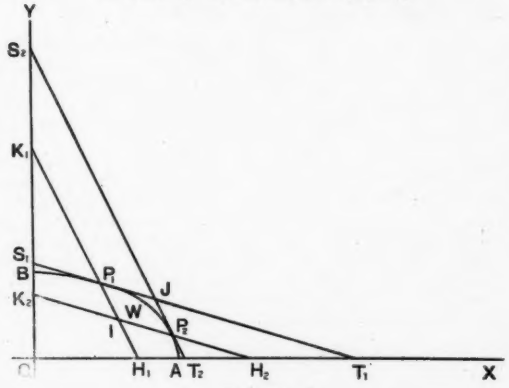


FIG. 18.

Hence the angles CH_1K_1 and CT_2S_2 are equal, and the lines H_1K_1, S_2T_2 are parallel.

Similarly the angles CH_2K_2 and CT_1S_1 are each equal to ϕ_2, and the lines H_2K_2, S_1T_1 are parallel.

Or, the tangent at either point of a Fagnano pair is parallel to the trammel line through the other.

29. As CD_1 is parallel to S_1T_1 (Fig. 15), it follows from the last article that it is also parallel to the trammel line through P_2, and the angle $A'CD_1=\phi_2$.

Similarly, CD_2 is parallel to the trammel line through P_1 and the angle $A'CD_2=\phi_1$.

It has been shown (Art. 5) that if the ordinate PN of any point P on an ellipse is produced to meet the auxiliary circle in Q and Q' (Figs. 2 and 4), the trammel line HK is parallel to CQ'.

Hence, if P_1N_1 and P_2N_2, the ordinates of the pair P_1 and P_2 (Fig. 19), are produced to meet the auxiliary circle, the first in Q_1 and Q'_1, the other

in Q_2 and Q'_2, then both CQ'_1 and CD_2 are parallel to the trammel line through P_1. Therefore CQ'_1 and CD_2 are in the same straight line.

Similarly CQ'_2 and CD_1 are in the same straight line.

From this the solution of the following is seen.

Given any point P on an ellipse to draw the diameter conjugate to that diameter which terminates in the Fagnano conjugate of P.

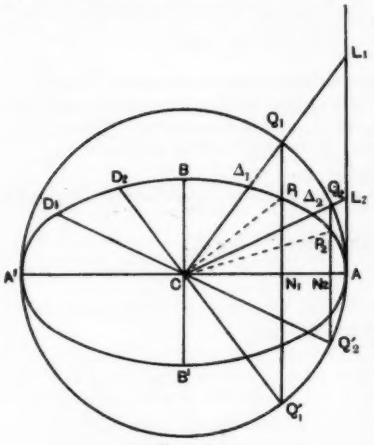


FIG. 19.

30. Further, let CQ_1 and CQ_2 cut the ellipse in Δ_1 and Δ_2.

Then by symmetry in the ellipse, $C\Delta_1 = CD_2$, or $C\Delta_1$ is equal to the semi-diameter conjugate to CP_2.

Similarly, $C\Delta_2$ equals the semi-diameter conjugate to CP_1.

Or, the part within an ellipse of the radius that bounds the eccentric angle of any given point on it, is equal to the semi-diameter conjugate to that diameter which terminates in the Fagnano conjugate of the given point.

31. Returning to Fig. 18, let H_1K_1 and H_2K_2 meet in I, and let S_1T_1 and S_2T_2 meet in J. Then P_1IP_2J is a parallelogram, whose angles P_1 and P_2 are each equal to $\phi_1 - \phi_2$.

When P_1 is at B, P_2 will be at A, and I at C. The parallelogram will then coincide with the rectangle contained by CA and CB.

As P_1 and P_2 approach one another, moving along the arc BA, the angles IP_1J and IP_2J become more and more acute, and the parallelogram diminishes, but obviously always contains the double point W. As the movement continues the parallelogram closes round W, and finally vanishes when P_1 and P_2 coincide in that point.

32. To find the area of the parallelogram P_1IP_2J, for any position of the pair P_1 and P_2.

The area of any parallelogram is the product of its two altitudes divided by the sine of one angle.

But it has been shown that the perpendiculars from C on to S_2T_2 and H_1K_1 are equal to CD_1 and $CD_1 \cos(\phi_1 - \phi_2)$ respectively.

Hence one altitude of the parallelogram

$$= CD_1 - CD_1 \cos(\phi_1 - \phi_2)$$

$$= 2 \,.\, CD_1 \,.\, \sin^2 \frac{\phi_1 - \phi_2}{2}.$$

Similarly, the other altitude

$$= 2 \,.\, CD_2 \,.\, \sin^2 \frac{\phi_1 - \phi_2}{2};$$

$$\therefore \text{ the area of } P_1IP_2J$$

$$= 4 \,.\, CD_1 \,.\, CD_2 \,.\, \sin^4 \frac{\phi_1 - \phi_2}{2} \operatorname{cosec}(\phi_1 - \phi_2)$$

$$= 2ab \sin^3 \frac{\phi_1 - \phi_2}{2} \sec \frac{\phi_1 - \phi_2}{2}.$$

33. By making use of other pairs of parallel lines, as described in previous articles, other parallelograms may be obtained in connection with a Fagnano pair P_1 and P_2.

For instance, it can be easily shown that if E is the point in the second quadrant symmetrical to P_2 in the first; then by drawing tangents at P_1 and E, and also the shorter trammel lines through these points, a parallelogram is obtained that is a minimum when P_1 and P_2 are at B and A, but a maximum when P_1 and P_2 coincide in W, and E therefore is at the Fagnano double point in the second quadrant.

Curvature at Fagnano Points.

34. It is clear from the parallelogram P_1IP_2J (Fig. 18) that the external angle between the tangents to the ellipse at P_1 and P_2 equals $\phi_1 - \phi_2$. But the angle Q_1CQ_2 (Fig. 17) equals $\phi_1 - \phi_2$; and, therefore, the same is true of the external angle between the tangents to the auxiliary circle at Q_1 and Q_2. That is, the total curvature of an arc of an ellipse terminated by a Fagnano pair of points is equal to the total curvature of the corresponding arc of the auxiliary circle.

35. Let ρ_1 and ρ_2 be the radii of curvature of the ellipse at P_1 and P_2.

Then by a well known formula

$$\rho_1 = \frac{CD_1^2}{CM_1} = \frac{CD_1^2}{CD_2},$$

also

$$\rho_2 = \frac{CD_2^2}{CD_1}.$$

Hence ρ_1, CD_1, CD_2 and ρ_2 are in geometrical progression; and

$$\rho_1\rho_2 = CD_1 \cdot CD_2 = ab.$$

It follows that the radius of curvature at $W = \sqrt{ab}$.

If for CD_1 and CD_2 are substituted their values given in Art. 26 (iii), we obtain

$$\rho_1^2 = ab \sin^3 2\phi_1 \operatorname{cosec}^3 2\phi_2,$$

$$\rho_2^2 = ab \sin^3 2\phi_2 \operatorname{cosec}^3 2\phi_1.$$

Fagnano Double Points on a Family of Ellipses.

36. When a line of constant length slides between fixed rectangular axes, its envelope is a four-cusped hypocycloid. This is a stock example in the Differential Calculus; but it may also be shown by elementary geometry.

If the different points on the sliding line are in succession regarded as carried points, a family of ellipses is obtained; and it is easy to show in an elementary manner that the envelope of this family is the same as the envelope of the sliding line, and that each ellipse of the family touches the envelope in its Fagnano double points.

For, when the trammel line HK slides between XX' and YY' (Figs. 5 and 6), the point of contact of any one of its positions with the envelope is the foot of the perpendicular let fall on to it from O, the instantaneous centre of rotation.

(See Lamb's *Infinitesimal Calculus*, Art. 163, Ex. 3 and 4.)

Hence, when HK is in such a position that P, a given carried point on it, is the foot of this perpendicular, then HK touches in P the envelope of its different positions.

But this happens when CZ (Fig. 9) is perpendicular to HK: in which case HK is also a tangent at P to the ellipse described by P, and the point of contact is a Fagnano double point on that ellipse.

Therefore the family of ellipses has the same envelope as the different positions of HK; and each complete ellipse of the family touches that envelope in its four Fagnano double points.

Short Proofs of Well-Known Formulae.

37. (i) In Fig. 11 suppose cp to be joined and e to be the middle point of pz. Then cp and cz are the lengths of a pair of semi-conjugate diameters in the ellipse; and, by varying c on the circle, cp and cz can be made to represent any pair of semi-conjugate diameters.

But $cp^2+cz^2=2ce^2+2ep^2$, which is constant as c moves;

$\therefore$ CP^2+CD^2 is constant.

(ii) A proof that the parallelogram contained by CP and CD is of constant area has been given incidentally in Arts. 23 and 24.

For it has been shown that in Fig. 14, $c_1z \,.\, c_1m=c_1z \,.\, c_2z=ab$;

$\therefore$ in Fig. 15, $CD_1 \,.\, CM_1=ab$.

(iii) In Fig. 7 draw the normal to the P-ellipse at P. This will be parallel to CZ. Suppose it meets CA in G and CB in G'.

Then $PG : CZ=PH : ZH$,

or $PG : CD=b : a$.

Also $PG' : CZ=PK : ZK$,

or $PG' : CD=a : b$. Percy J. Harding.

EUCLID'S POSTULATE AS A PROPERTY OF MATTER.

The time has long passed when the teacher of elementary geometry, however elementary, can efficiently exercise his individuality in devising improved methods of exposition without taking some cognisance, however small, of the study of non-Euclidean geometry which has grown up out of the failure to prove what may be described as "Euclid's Parallel Postulate."

A paper has recently been published in the *Proceedings* of the Edinburgh Mathematical Society by Prof. Carslaw, of Sydney, N.S.W., containing an elementary interpretation of the Bolyai-Lobatschewsky geometry, which is so simple and so easily intelligible that I strongly recommend every teacher of mathematics or science to read it. It does away altogether with the necessity of forming conceptions of space which appear contradictory to experience, and shows the possibility of constructing a non-Euclidean geometry in ordinary Euclidean space.

The present article is, at least as far as the first part is concerned, entirely based on Prof. Carslaw's paper, of which an attempt at a brief resumé is now proposed.

The simplest kind of *two dimensional* geometry, which is essentially non-Euclidean, is *ordinary* spherical geometry, and is sufficiently well known to need no comment. If great circles on a sphere take the place of straight lines in a plane, we have a geometry in which the three angles of a triangle are together greater than two right angles, in which all great circles intersect in two points, and in which therefore parallelism is impossible, since through a given point it is impossible to draw a great circle which does not cut a given great circle not passing through that point.

The lines which in a new geometry of this kind take the place of straight lines in ordinary Euclidean geometry are called by Carslaw "ideal lines." Their intersections are "ideal points." At the same time, while Euclidean straight lines intersect in only one point, ideal lines may cut in two points. So if we read Carslaw aright an ideal point might be two points; but in any case this does not affect his arguments. Applying these notions to the sphere, ideal lines in the case above considered are great circles.

Now, in the first place, Carslaw points out that the geometry of the sphere is not *necessarily* non-Euclidean, but that this depends entirely on what are defined to be *ideal lines*. This is the reason why we have spoken above of *ordinary* spherical geometry. Even in this connection several points might be raised which would take too long to discuss here.

A Euclidean system of spherical geometry can be obtained from Euclidean plane geometry by the simple process of inversion. The ideal lines of such a system will be circles through the centre of inversion. This point will represent infinity on the plane. Prof. Carslaw shows how "ideal distances" and "ideal displacements" can be defined on the sphere in such a way as to give rise to geometrical properties identical in substance with those known to hold on the corresponding plane, and therefore in accordance with Euclid's postulate.

Having thus established the fact that a system of Euclidean geometry can be built up on a sphere, Carslaw now propounds a system of non-Euclidean geometry applicable to a plane, or, for three dimensions, to ordinary Euclidean space; we need only consider here the two-dimensional problem.

In this system *ideal lines* are circles cutting at right angles a certain fixed circle—the *fundamental circle*. An *ideal point* represents the two points, one inside and one outside the fundamental circle—in which two such circles intersect. Through any given point A there will be two such ideal lines touching any ideal line UV on the fundamental circle. These will represent the two parallels through a given point to a given line of the Bolyai-Lobatschewsky geometry. They will separate those ideal lines through A which do from those which do not cut the given ideal line in real points. The fundamental circle corresponds to infinity in plane geometry, and "ideal lengths" and "ideal displacements" are so defined that in any displacement the length of any segment is unaltered, and, further, the ideal distance of any point from the fundamental circle becomes infinite.

The parts of any figure inside and outside the fundamental circle are the inverses of each other, and only one of these need be drawn in proving any geometrical property. But the distance between inside and outside points contains an imaginary logarithmic constant.

Carslaw shows that the geometry of a system so defined satisfies all the properties of hyperbolic geometry as the Bolyai-Lobatschewsky system is called. Ideal angles in his system appear to be the same as the ordinary geometrical angles at which the circles cut. This arises from the fact that *ideal displacements* are defined as the result of successive inversions with respect to circles of the system just as in plane geometry a displacement can be produced by two reflections in a straight line. It is shown, further, that in this geometry—as in the geometry of great circles on a sphere, similar triangles are impossible.

By showing in this way that a non-Euclidean geometry can be built up in ordinary Euclidean space, Carslaw shows the impossibility of *proving* Euclid's parallel postulate as a property of *space*. For he has shown that the same kind of space which possesses a Euclidean geometry necessarily possesses a hyperbolic geometry as well.

I now venture to take up the argument where Carslaw leaves it, and discuss the view that Euclid's parallel postulate is thus shown to be a property of *matter* and not of *space*.

The definition of ideal lengths and ideal displacements naturally leads to the concept of an *ideal rigid body* defined as an assemblage of points capable of displacement in such a way that the ideal distance between any two specified points remains constant. And that which fixes such an assemblage of points may be defined as ideal solid matter.

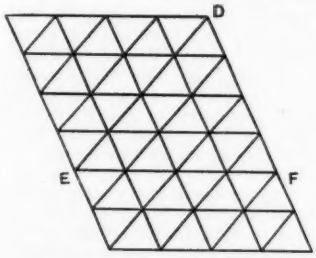
It thus appears definitely established, unless I am very greatly mis-

taken, that ordinary Euclidean space is capable of containing ideal matter in the solid—or other—states conforming to the properties of Carslaw's non-Euclidean system and having any definite fundamental sphere whatever. The fundamental sphere is of course the three dimensional equivalent of the common orthogonal circle, or fundamental circle of the plane system considered above, as is more fully discussed by Carslaw.

If, on the other hand, it were attempted to make a Euclidean rigid body undergo a displacement satisfying a Carslaw condition of rigidity, strains would be set up. But as no body is perfectly rigid, the only result would be to produce stresses in the body, and if the displacement were small in comparison with the radius of the fundamental sphere, these stresses would be small. If this contention is correct, does not Euclid's postulate reduce to an experimental result in the Theory of Elasticity? At the same time, it should be pointed out that for a displacement comparable with the radius of the fundamental sphere the strains set up either in the Euclidean or the Carslaw "ideal" body would become great if the displacement did not conform to the geometry of the matter of which the body was formed.

These considerations raise the third question put forward in this paper: If Euclid's postulate is thus proved to be a property of matter, what is the best way of teaching those parts of geometry which depend directly on its truth, so that the material aspect may be put most prominently before the learner?

Now there are few starting points more suggestive and instructive than the tessellated pavement formed of triangular tiles all of the same size and shape as shown in the figure (this is no new idea). Not only is it possible to read off from the diagram a large number of properties of parallel straight lines and parallelograms proved in Euclid, Book I., but a number of properties of similar triangles whose sides are



relatively commensurable can be deduced by considering two large triangles, each built up of the smaller ones. For example, that the areas of two such similar triangles are proportional to the squares of the lengths of their homologous sides can be verified at once by counting the number of small triangles which go to make one such large triangle as *DEF* in the figure.

The property of matter which is assumed in this diagram is that a pavement formed of equal triangular tiles can be extended indefinitely without unduly straining the tiles, without anywhere varying their order of arrangement, and without leaving any blank spaces between them.

If we were to try to cover the whole surface of the terrestrial globe with the same triangular tiles, we could not do so without breaking the regularity of the arrangement somewhere, or leaving some gaps.

If we were to cover a sphere with a network of "ideal lines" obtained by the inversion of our figure, as in Carslaw's Euclidean system of spherical geometry, we should have a possible tessellated pavement covering the sphere, but the triangles could only be regarded as all of the same size and shape by a convention regarding ideal length which would not accord with the properties of tiles formed of ordinary matter. Towards the origin of inversion, the triangles would appear to us to get indefinitely small as the "ideal length" of their sides departed more and more from their true length, and it would become increasingly difficult to strain material tiles to the right shape.

It is not difficult to see that in order that this tessellated pavement may be extended indefinitely in all directions, the matter of which the tiles are made must conform to Euclid's parallel postulate. If it does not do so there must necessarily be breaks in the regularity of the arrangement. Furthermore, it is obviously impossible to make any continuous pavement at all out of triangular tiles of the same size and shape if the sum of their angles differ a little from two right angles. Even if we try to make a large triangle out of four smaller triangles, we shall fail, for the sides of the figure so constructed will have a bend in them at the points where the triangles join, *i.e.* at what ought to be the middle points of the sides.

In conclusion, I suppose it may now be regarded as proved that Euclid's parallel postulate is to be considered as a property of matter rather than of space, on the ground that it is possible to establish non-Euclidean geometries in ordinary Euclidean space by assuming suitable definitions of distance and displacement. The only thing which experimental evidence can show is that such definitions may be incompatible with the conceptions of distance and displacement derived from our experience of material bodies. And if this be so, the bearing of these conclusions on the teaching of geometry is obvious. G. H. Bryan.

THE THEORY OF THE POLAR PLANIMETER: AN APPLICATION OF THE METHOD OF BARRIERS.

Many instruments have been devised for mechanically recording the area of a closed curve when a tracing point is carried round its boundary, and, of these, probably the best known is the Amsler Planimeter invented in 1854 by Professor J. Amsler of Schaffhausen.

It belongs to the class known as polar planimeters, so called because the tracing arm turns about a fixed centre or pole.

The theory of the instrument has been given in various ways by different writers, but perhaps the neatest explanation is that given by Professor Henrici in the *British Association Report*, 1894. As this is probably pretty well known, only the briefest resumé will be given here in order to make the extension of it and the present article readily intelligible. The explanation depends on the general theorem that the area, reckoned algebraically, swept out by a line of fixed length in a complete cycle, is equal to the difference of the areas described by its ends. In the Amsler Planimeter, one end of the tracing arm moves over the circumference of the circle described by the end of the pole arm while the tracing point at the other end describes the boundary of the area.

Hence, if the pole be fixed *outside* the area, the tracing arm AB (Fig. 1) will perform a complete cycle without, on the whole, rotating, and the area swept out by it will be equal to that described by the tracing point B, viz. BCB_1DB, since the other end A moves to and fro along the arc Aa_1A_1aA, and the area described by it is therefore nil.

The cancelling of positive and negative areas is shown by the cross-hatching.

If now the motion of the bar be resolved into motions of translation and rotation, it is obvious that in the cycle first described the latter cancel together with the rollings of the wheel due to them, and that therefore the area swept out is due to translation only, and is, consequently, proportional to the rolling of the wheel. Hence the latter, whatever its position on the tracing arm, records the area of the curve.

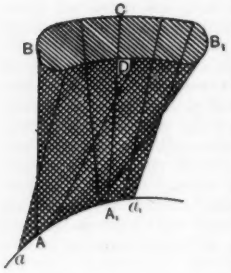


FIG. 1.

If, however, the pole be fixed *inside* the area, the simple result just obtained will no longer hold, for the arm will now make a complete rotation, and the areas due to rotation will not therefore cancel.

In this case the rolling of the wheel is a function of its position on the arm, and is not simply proportional to the area described by the point.

Prof. Henrici shows that the area required is obtained from the reading of the wheel by adding the corresponding area to, or subtracting it from, the area of the datum circle according as the latter is inside or outside the curve. This is the circle traced out by the point when the arm is held so that the plane of the rim of the wheel passes through the pole *O*. Since in this position no rolling can take place, the area recorded is zero, and the circle described has therefore been variously called the zero, base or datum circle.

It would be almost expected, *a priori*, that the area recorded in this case would be the difference of the areas of curve and datum circle. It is the object of this article to show that this is but a particular case of that just discussed in which the pole is outside the area. Three cases may be considered according as the curve lies (1) entirely outside, (2) entirely inside, or (3) partly outside and partly inside the datum circle.

Considering the first of these, suppose the tracing point to be carried once round the curve *PQRP* (Fig. 2) in the positive (clockwise) direction, and the reading to be noted. This reading will be the same as it would be if the point had been carried round the curve *PQRP*, then along any arbitrary path *PS* to any point *S* on the datum circle, round the latter in the positive (this time, anti-clockwise) direction *STUS*, and finally back along *SP* to the starting point *P*. For no rolling will take place while the point is describing the datum circle, and the rolling during the motion from *P* to *S* will be cancelled by that occurring on the return motion from *S* to *P*. The area recorded is here obviously that of the shaded figure *PQRPSTUSP*, for the point has been carried once round its boundary, the pole *O* being outside. In fact the figure may be regarded as that of a long narrow area bent round so as to completely envelope the pole *O* which is outside it, the two ends becoming coincident in the line *PS*. Hence, the ordinary method of using the instrument consists in suppressing those operations which cause no rolling of the wheel.

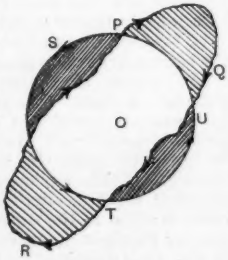


FIG. 2.

By similar reasoning, if the curve lie entirely inside the datum circle, as in Fig. 3, the area recorded will be again the difference of areas of curve and datum circle. But here it must be noted that if the curve be described

in the usual clockwise direction, the shaded area will be described in the negative direction, and *vice versa*, as will be shown by the wheel. This is indicated by the shading.

Finally, if the curve lie partly outside and partly inside the datum circle, as in Fig. 4, it follows from the previous cases that the wheel will record the

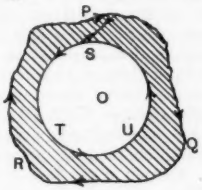


FIG. 3.

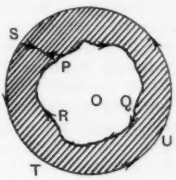


FIG. 4.

excess of the areas lying outside over those lying inside the datum circle. This case is really that of a looped curve bent round the pole O, the area obtained being, as usual, the difference of the sums of the odd and of the even loops.

The device just employed is merely an application of the method of barriers by which any n-ply connected region may be made simply connected by drawing $n-1$ arbitrary barriers. The contour PQR and the datum circle form a doubly connected region, which is made simply connected by drawing one arbitrary barrier PS.

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Technica' College, Bradford.

MATHEMATICS OR DRUDGERY?

MR. C. S. JACKSON, in the course of his remarks on the Report of the Joint Committee upon the "Co-ordination of the Teaching of Science and Mathematics," refused to admit the plotting of $y=\sqrt{\cdot 5-\frac{1}{10\sqrt{x}}}$ into the realm of Mathematics, and although he did not state his reason for so doing, it is not difficult to guess. In the formula quoted, Mr. Jackson has, I think, only seen a double radical not lending itself readily to calculation; and he looks down on the plotting with contempt as being an *aimless manipulation*, which feeling must be shared by anyone having true mathematical knowledge or instinct.

The ascent of a long flight of steps in order to get a glimpse of the view from the top of a tower is indispensible, but is at the same time, only a material hindrance keeping one back when, tempted by peeps obtained here and there through narrow windows, one longs for the open summit. Similarly, manipulation in mathematics is necessary, although hampering, keeping back the eager anticipating mind until it can bear it no longer, and Newton needs to beg from a friend the performance of the calculations to which, in his emotion, he felt incapable of curbing his soaring genius, dazzled by the vastness of its own flight. The climbing *per se* is but a means to an end, and the performance of it for its own sake would be absurd, except in the case of the child for whom it is necessary to exercise his limbs, and in the case in which the climbing is of such a difficult nature as to be in itself an operation requiring great skill. It is exactly the same in mathematics, where mere manipulation,

as an aim in itself, should not be introduced except at the very beginning; but beyond that stage should serve simply as an auxiliary.

The various mathematical processes may be said to consist of Manipulation and Principles. The former we have eliminated from the realm of mathematical science; remains the latter constituting the Philosophy of Mathematics. The question here presents itself: is this philosophy or meta-mathematics (if we may so call it) to be the sole sphere of the true mathematician? I venture to suggest that other criteria exist, enabling one to discriminate between truly mathematical manipulation and that aimless juggling with symbols and figures which is abhorrent to the spirit of true mathematics; and first among these stand *method* and *utility*. No less a mathematician than Sir George Airy attributed his mathematical skill to a high appreciation of *method*, and spoke of mathematics as nothing more than method carried to a considerable extent. At the meeting, Professor H. H. Turner, a no less able mathematician and astronomer, led us to infer that *utility* is the supreme test. For if one may tell a mathematician by his ability to deal with a camel driver shamming sick, may we not discern true mathematics from false by its *purpose*?

It must be admitted that, when we condemn manipulation as drudgery, it is on the ground of being aimless. There are few more tedious operations than calculating, by the method of least squares, the probable error of a set of constants obtained by observation; and the calculation cannot be said to derive interest from its intricacy. Yet an observer would never call it drudgery, and I, personally, think that it is one of the most interesting parts of the work of reducing one's observations, without which this work is neither complete nor useful.

A mathematician labelled as unmathematical the plotting of the formula above, because its plotting did not involve any fundamental principle of far-reaching importance or of wide-world application. Had he been confronted with such an expression as

$$y=2x^3-6x+15; \quad \text{or} \quad y=x^3+3x+2,$$

the knowledge that it involves principles governing the number of solutions of an equation; the number of equal or of imaginary solutions; the number of intersections, real or imaginary, of the graph with a graph of the form $y=mx+n$, etc., would have probably made him consider its plotting as mathematics. Yet, after all, it does not embody, but only illustrates, principles that need independent proof of a stricter nature. Although mathematics, the graph of such a function will not be a means to an end, and can hardly lay claim to a very definite purpose, nor does it enforce strict method, as the calculations can be jotted down desultorily as the plotting proceeds. Not so with a formula of the form $y=\sqrt{\cdot 5-\frac{1}{10\sqrt{x}}}$, especially if the numerical constants are not so simple as those given here. The student who starts it in a happy-go-lucky fashion will readily endorse Mr. Jackson's opinion, and provide suitable epithets to replace the one irretrievably lost through his reserve. Let him, however, arrange his work intelligently, and prepare himself to tabulate x, $10\sqrt{x}$, $\frac{1}{10\sqrt{x}}$, $\cdot 5-\frac{1}{10\sqrt{x}}$, and y, and in a few shifts of the slide rule, column after column will rise, and a wearisome computation will be transformed into a rapidly growing table of useful data, from the last column of which the Spanish Modula in its correct shape will emerge, a tangible triumph of method. For, let the student draw the graph, and lo! it is not an abstract thing he has traced on his squared paper; it is the exact

outline of a certain solid of revolution (used on the Canal of Isabella II. near Madrid) for allowing a constant flow of water to be discharged despite variations of head. What could be more "elegant" and "useful," to use Mr. Jackson's own words? Possibly I may have misunderstood him, but I cannot agree with him that a manipulation which trains the mind to *method*, and which leads in such a *direct* manner to such an *elegant* solution of an important *practical* problem is likely to shame mathematics. That otherwise we agree is evident from the fact that, in the paper he referred to, I condemned the plotting of that very same formula if made in the wrong spirit and turned into an aimless manipulation.

To sum up: drudgery can never claim recognition as being related to Mathematical Science, while the right of manipulation to be reckoned within its sphere may be tested by the criteria of *method, utility*, and *elegance*.

M. E. J. GHEURY.

I agree with Mr. Gheury's views, and think that he has expressed very clearly what I intended to convey, though his speech, at any rate in the necessarily brief report of it, may have failed to make it clear. A common fault of elementary text-books has been to give a bit of heavy manipulation without giving any hint of the object, whether of result or of method, which justified the drudgery involved. It was this practice which I intended to denounce and which Mr. Gheury agrees with me in denouncing. C. S. J.

ANSWERS TO QUERIES.

[68, p. 189, vol. v.] If the pairs of opposite sides of a quadrilateral are a, a' and b, b' and the diagonals are c, c', then the condition for contact of the circles on c, c' as diameters is

$$a^2+a'^2+b^2+b'^2=2(c^2+c'^2\pm cc') \quad \text{.....(1)}$$

$$\left(\text{For distance between mid-pts. of } c, c' \text{ is } \frac{a^2+a'^2+b^2+b'^2-c^2-c'^2}{4}.\right)$$

The question is whether the values of c, c' given by this condition and

$$\begin{aligned} &c^2c'^2(a^2+a'^2+b^2+b'^2-c^2-c'^2) \\ &+a^2a'^2(b^2+b'^2+c^2+c'^2-a^2-a'^2) \\ &+b^2b'^2(a^2+a'^2+c^2+c'^2-b^2-b'^2) \\ &-a^2b'^2c^2-a'^2b^2c^2-a^2b^2c'^2-a'^2b'^2c'^2=0 \end{aligned} \quad \text{.....(2)}$$

are real.

A sextic equation for c/c' can be formed from (1) and (2). If it has a real root λ, then, taking upper sign in (1),

$$c=\lambda c'=\lambda\sqrt{\frac{a^2+a'^2+b^2+b'^2}{2(1+\lambda+\lambda^2)}}, \text{ a real quantity,}$$

and these values of c, c' are real and satisfy (2), so that the quadrilateral exists.

Hence the condition that with given sides a, b, a', b' it may be possible for the circles to touch is the condition of reality of roots of the sextic for c/c'.

A. C. DIXON.

[69, p. 330, vol. v.] The problem can be reduced to the discovery of a chord APQ drawn from a fixed point on a circle $BACQ$ and cutting a fixed chord BC in P such that PQ is of given length.

If $AC=b$, $AB=a$, $BC=c$, $CP=x$, and $PQ=k$,

$$AP^2=b^2+x^2-2bx\cos C,$$

$$PQ^2=k^2,$$

$$AP.PQ=BP.PC=x(a-x),$$

and $$k^2(b^2+x^2-2bx\cos C)=x^2(a-x)^2,$$

a biquadratic equation, which can only be solved geometrically when $b=c$, and it reduces therefore to

$$k^2(b^2+x^2-ax)=(x^2-ax)^2.$$

The geometrical treatment of the problems in Queries 69 and 70 involves some interesting relations between the lines and triangle of the figures.

J. E. A. STEGGALL.

(See also *Mathematics from Ed. Times*, xiv. 1908, Qu. 16415.)

[69, p. 330, vol. v.] *Through a given point to draw a straight line of given length terminated by two given straight lines.*

This is impossible by Euclid's methods.

For, let XOX', YOY' be the straight lines, P the given point, supposed situated in the angle XOY. Then, if the given length is greater than the shortest line which can be drawn through P terminated by the given lines, then two such lines can be drawn in the quadrant XOY, and evidently one such line, which is always real, can be drawn in each of the quadrants XOY', $X'OY$; hence there are four solutions, and the problem depends on a biquadratic equation.

Analytically, the solution is given by

$$\rho_1\sec(\theta-\alpha)-\rho_2\sec(\theta-\beta)=a,$$

where the point is taken as origin, the equations of the two lines being $\psi=\rho_1\sec(\theta-\alpha)$, $r=\rho_2\sec(\theta-\beta)$.

A simpler solution by "conics" is obtained by constructing the loci of the middle points (a) of a line through the given point terminated by the given lines (this is a hyperbola), (b) of a line of given length terminated by the given lines (this is an ellipse, or if the lines are at right angles, a circle). The intersections of these conics give the middle points of the required lines.

G. H. BRYAN.

[70, p. 330, vol. v.] The problem can be reduced to the discovery of a chord APQ drawn from a fixed point on a circle $BACQ$, and cutting a fixed chord BC in P such that PQ is a maximum. This leads to a cubic equation, and the chord therefore cannot be obtained by ordinary geometrical methods, except when the point A bisects the arc BAC.

J. E. A. STEGGALL.

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REVIEWS.

A Text-Book of Differential Calculus. By GANESH PRASAD, B.A. (Cambridge), D.Sc. (Allahabad). Pp. 161. (Longmans, Green, & Co. 1909.)

This work deals with the elements of the subject in a more systematic way than most of the older books. It is intended for beginners, and is designed to meet the requirements of Part I. of the Cambridge Mathematical Tripos, and of the Examinations for the B.A. and B.Sc. degrees of Indian Universities. Unlike most modern elementary text-books on geometry or the calculus, it does not aim at making the subject intelligible at the expense of rigidity to students who take up mathematics for merely practical purposes; for all the definitions and proofs

are strictly rigorous. At the same time certain conceptions are stated as self-evident, *e.g.* in Rolle's theorem, with a note to the effect that a more advanced investigation may be found in some book on Functions of a real variable.

Proceeding from accurate definitions of variable, function, limit, continuity, and differential coefficient, for real magnitudes, the order followed, broadly speaking, is to teach how to differentiate the ordinary known functions, and to apply the results to the differential geometry of a curve. Thus the tangent, normal, asymptote, curvature, pedal and inverse, etc., are dealt with, concluding with some useful hints on curve tracing. The rest of the book investigates successive differentiation, Rolle's theorem, Taylor's theorem, maxima and minima, and indeterminate forms, with a few suggestive notes at the end, which are designed to indicate some of the difficulties of a thoroughly satisfactory proof of the theorem of the mean.

Apparently a second volume is to follow which will deal with partial differentiation, double limits, and the theory of multiple points.

The purely differential parts of the geometry are good: results are given systematically and clearly, without undue length or superabundant detail. One would, however, like to see a more elegant method of dealing with asymptotes. They are introduced with much brevity, but might have received easier treatment. As multiple points are not considered, the short reference to asymptotes seems a little out of place in a book which is otherwise purely a differential calculus. Perhaps they will be found more fully worked out in Vol. II.

The later parts of the book are well done, particularly the theory of maxima and minima. No geometrical aid is sought; it is defined and demonstrated by analysis.

There is a curious intentional omission of any ideas of orders of small or great quantities. The author explains that they are not necessary for a beginner. Perhaps not; but as every schoolboy makes use of them in working out arithmetical examples correct to three or four decimal places, one would think that the explanation of the underlying principle might well be introduced into a first course on the Calculus.

A Text-Book of Integral Calculus. By GANESH PRASAD, B.A., D.Sc. Pp. 241. (Longmans.)

The aim of this book, like that of the corresponding Differential Calculus, is to be simple but rigorous, and the result is certainly good. To effect this, integration is defined as the inverse operation of differentiation, and five chapters are devoted to working out all possible indefinite integrals in terms of the simpler known functions. The treatment is not novel, but it is practical and systematic. Then follows an instructive chapter on the definition of a definite integral as the limit of a sum, in which no attempt is made to convince the beginner that an elementary proof of the fundamental theorem can account for all possible cases, *e.g.* when one limit of integration is infinite. Where a strict proof involves ideas too advanced for an elementary text book, there is a note of explanation added in the appendix as a sort of introduction to higher analysis. Thus there are references to integration term by term of infinite series, and the use of Fourier's Series.

The rest of the book makes wide applications of the Calculus to geometry and mechanics, with a short introduction to differential equations. Thus lengths of curves, areas of plane and curved surfaces, volumes, moments of inertia, centres of gravity and pressure, and so on, are investigated briefly and clearly, and the results are printed in good clear type, so as to make the book useful and easy for reference.

In both these two works there is a good selection of examples taken from recent examination papers at London and Cambridge Universities and elsewhere.

Calculus made Easy. By F. R. S. Pp. 178. (Macmillan & Co.)

"Being a very-simplest introduction to those beautiful methods of reckoning which are generally called by the terrifying names of the Differential Calculus and the Integral Calculus." So reads the title page. And it must be confessed that since the days of Lewis Carroll very little has been done to try and interest the lay mind in the mysteries of mathematical symbols. However, this is an attempt to explain the difficulties of the Calculus in an easy conversational way

to engineering students of a non-mathematical turn of mind. To be able to differentiate and integrate such functions as have value in practical mechanics and engineering, one must have *some* idea of the meaning of these processes. As long as they are taught on a purely mathematical foundation they are unintelligible to the man of a purely practical turn of mind. But when commonsense is used, and when it is explained that dx means 'a little bit of x,' and Sdx means the sum of all the little bits of x, and all the refinements of mathematical phraseology are replaced by their simplest paraphrases, there is a hope that more general interest will be aroused in the subject. No one objects to teaching a child long division because the validity of the method has not been demonstrated to him! Surely, then, it is equally right to make the Calculus easy and interesting to those whose interest does not prompt them to read Theory of Functions of a Real Variable. This is what the author tries to do. H. W. TURNBULL.

The Theory of Elementary Trigonometry. By D. K. PICKEN, M.A. Pp. 48. 1910. (Whitcomb and Tombs, Wellington, N.Z.)

In this little book, which may some day be a typographical curiosity as an early and very creditable specimen of mathematical printing in New Zealand, the author's aim is to present the subject in a manner suitable to beginners of mature intelligence.

The book deals only with the definitions of the trigonometric functions, their validity for angles of any magnitude, the addition theorems for the sine and cosine, and the geometry of the plane triangle.

The author has succeeded very happily in indicating that the construction and solution of trigonometric identities is not so arbitrary or so haphazard a proceeding as sometimes appears.

Elements of Plane and Spherical Trigonometry. By D. A. ROTHROCK, Ph.D., Professor of Mathematics, Indiana University. Pp. 147, 99. Price 6s. net. 1910. (Macmillan Company, New York.)

This book, which a binder eccentrically terms a $\frac{\text{Trigonom}}{\text{etry}}$, contains a sketch of plane and spherical trigonometry, followed by neatly printed and well designed tables of logarithms to five places, of four-figure numbers, logarithms of trigonometrical functions 0 to 3′ by 1″, 0″ to 2° by 10″, 0′ to 90° by 1′ and a four-figure table of natural trigonometric functions by minutes. Secants and cosecants are omitted, an example which seems likely to be followed when working with co-logarithms becomes familiar. The text, though concise and plain, is less noteworthy, and the treatment of higher trigonometry makes no pretension to rigour.

At p. 97, the equation $x=a+b\sin x$ is solved by graphing $y=x-a$ and $y=b\sin x$. With a slide rule this would no doubt be just as easy as graphing $y=\frac{x-a}{b}$ and $y=\sin x$, but the slide rule is not mentioned. C. S. J.

The Trisection of the Angle by Plane Geometry, verified by Trigonometry and Concrete Examples. By J. WHITEFORD. 1911. (Bowes and Bowes, Cambridge.)

Dr. Whiteford has found a formula for the third part of a given angle, and applies it to fifty selected cases. The full working is duly set forth on pages of good paper, about a foot long and 9½ in. broad, lines spaced at about five to the inch. This stately marshalling of the arithmetical procedure is worthy of a better cause than the computation of sines and cosines to seven figure accuracy. Unfortunately, Dr. Whiteford seems to think that by attaining this degree of accuracy he has solved the celebrated problem to the solution of which so many minds have set themselves in the ages past. He seems to be quite aware of the fact that the problem has been classed among those that are insoluble, and quotes from De Morgan to that effect. We fear that he may continue to hug his comfortable delusion in spite of all that can be said, but we would ask him seriously to consider p. 15 in Messrs. Beman and Smith's translation of Klein's *Famous Problems of Elementary Geometry* (Messrs. Ginn) before proceeding further with his investigations.

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